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ON INTEGRABILITY OF THE GIL'DEN-MESHCHERSKII PROBLEM*

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It is shown that the mathematical laws of mass change under which the classical Gil'den-Meshcherskii nonstationary two-body problem is reduced to its autonomous form /1-3/, remain invariant with respect to the symmetry group admitted by the problem, and consequently possess a physical character. It is also shown that the question of the integrability of the problem under consideration can be reduced, for any law of mass change, to the problem of integrability, when the mass changes according to the generalized Eddington-Jeans law with the index $1 \leqslant \nu \leqslant 3$.

In /1-3/ the Gil'den-Meshcherskii problem of two bodies of variable mass was considered, the problem described by the equation of the type

$$\mathbf{r}^{\cdot\cdot} = -\mu(t)\frac{\mathbf{r}}{r^3}, \quad \mathbf{r} = (x, y), \quad r = \sqrt{x^2 + y^2}, \quad (^{\cdot}) = d/dt \tag{1}$$

where $\mu(t)$ is a certain function of time. All laws of mass change $\mu(t)$ were found for which the problem (1) is reduced, using the Kummer-Liouville transformation

$$\mathbf{r} = \mathbf{v}(t)\,\boldsymbol{\rho}, \; d\tau = u(t)\,dt, \; \boldsymbol{\rho} = (\xi, \eta) \tag{2}$$

where $v(t)$ and $u(t)$ are sufficiently smooth functions, reduced to the autonomous form

$$d^2\boldsymbol{\rho}/d\tau^2 + b_1\,d\boldsymbol{\rho}/d\tau + b_0\boldsymbol{\rho} = -\mu_0\boldsymbol{\rho}\rho^{-3}, \quad \rho = \sqrt{\xi^2 + \eta^2} \tag{3}$$

(b_0 and μ_0 are real constants, and b_1 can be a real or purely imaginary constant). The laws obtained satisfy the integrodifferential equation

$$\mu^{\cdot\cdot} - 2\mu^{-1}\mu^{\cdot 2} + \frac{2b_1^2 + b_0}{k + 9b_1^2}\mu^5\left(\int_{t_0}^{t}\mu^2\,dt\right)^{-2} = 0 \quad \begin{pmatrix} k = 1, \; b_1 = 0 \\ k = 0, \; b_1 \neq 0 \end{pmatrix} \tag{4}$$

and can also be expressed in the form of finite equations (see /3/, formulas (4.1)-(4.5)). Equation (4), together with the formulas (4.1)-(4.5) of /3/ in combination, remain invariant relative to the symmetry group admitted by the problem (1). Indeed, let us obtain a transformation of the type (2), transforming the problem (1) into itself, i.e. to the form

$$d^2\boldsymbol{\rho}/d\tau^2 = -\mu_1(\tau)\boldsymbol{\rho}\rho^{-3} \tag{5}$$

where $\mu(t)$ and $\mu_1(t)$ must satisfy one and the same differential or integrodifferential equation, but need not coincide.

Lemma. The necessary and sufficient condition for the problem (1) to be reducible to (5) by means of the transformation (2) is, that (2) has the form of (6):

$$\mathbf{r} = (\alpha t + \beta)\,\boldsymbol{\rho}, \; d\tau = (\alpha t + \beta)^{-2}dt \quad (\tau = (\alpha_1 t + \beta_1)\,(\alpha t + \beta)) \tag{6}$$

where $\alpha_1\beta - \alpha\beta_1 \neq 0$. Here (1) admits a one-parameter Lie group G_1 with the generator X:

$$G_1: t^1 = \frac{(a\alpha\beta + 1)\,t + a\beta^2}{1 - a\alpha\,(\alpha t + \beta)}, \quad x^1 = \frac{x}{1 - a\alpha\,(\alpha t + \beta)}, \quad y^1 = \frac{y}{1 - a\alpha\,(\alpha t + \beta)} \tag{7}$$

$$X = (\alpha t + \beta)^2\frac{\partial}{\partial t} + \alpha\,(\alpha t + \beta)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

The results can be verified directly, or proved using /3/ and the theory of Lie groups. Omitting the mathematical procedures, we can arrive at the following theorem.

Theorem 1. The necessary and sufficient condition for the problem (1) to be invariant with respect to the transformation (6), (7) is, that the equation (4) be invariant with respect to the transformation

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$$\mu = (\alpha t + \beta)^{-1}\mu_1,\ d\tau = (\alpha t + \beta)^{-2}dt \tag{8}$$

i.e. that it admits the one-parameter Lie group G_1 with the generator M:

$$t^1 = \frac{(a\alpha\beta + 1)\,t + a\beta^2}{1 - a\alpha\,(\alpha t + \beta)},\quad \mu^1 = \mu\,[1 - a\alpha\,(\alpha t + \beta)] \tag{9}$$

$$M = (\alpha t + \beta)^2 \frac{\partial}{\partial t} - \alpha\,(\alpha t + \beta)\,\mu\,\frac{\partial}{\partial \mu}$$

Thus the set of integral curves of equation (4) describing the evolution of mass is invariant in the whole with respect to the transformation (8), (9). But here the problem (1) admits the symmetry group (7), therefore the law (4) and the corresponding fininte equations belonging to the set have a physical character.

Note. The Eddington-Jeans law (second Meshcherskii law) $\mu^{\cdot} = -k\mu^3$ $(\mu = (\alpha t + \beta)^{-1/2})$ and the experimental law $\mu^{\cdot} = -k\mu$ $(\mu = \mu_0 e^{-kt})$ are both invariant with respect to the transformation (8), (9), while the generalized Eddington — Jeans law (ν arbitrary)

$$\mu^{\cdot} = -k\mu^{\nu} \tag{10}$$

with the exception of the above cases of $\nu = 1$ and $\nu = 3$, is not invariant with respect to specified transformation. Nevertheless the mathematical law (10) plays a fundamental part in the problem of integrating (1), (4).

Theorem 2. When $1 \leqslant \nu \leqslant 3$, integration of the problem (1), (4) reduces to integrating the problem (1), (10).

Scheme of proof. We apply to the set of integral curves of equation (4) described by the already mentioned formulas (4.1)—(4.5) of /3/, the transformation of the type (8), (9). As a result the families of curves (1), (2) will be transformed into (4), the family (3) into (5) and the family (4) into itself. Thus, irrespective of the law of mass change in the problem (1), (4), (10) will represent the law of mass change for the tranformed problem (5). Now it remains to utilize the result of /4/ which states that integration of the problem (1), (10) for ν arbitrary can be reduced to integrating (1), (10) but already for $1 \leqslant \nu \leqslant 3$.

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REFERENCES

1. BERKOVICH L.M. and GEL'FGAT B.E., Study of certain nonstationary problems of celestial mechanics using the transformation methods. In coll. Problemy analiticheskoi mekhaniki, teorii ustoichivosti i upravleniia, Moscow, NAUKA, 1975.
2. BERKOVICH L.M., Gil'den—Meshcherskii problem and laws of mass change. Dokl. Akad.Nauk SSSR, Vol.250, No.5, 1980.
3. BERKOVICH L.M., Transformation of the Gil'den—Meshcherskii problem to its stationary form and the laws of mass change. PMM Vol.44, No.2, 1980.
4. RADZIEVSKII V.V. and GEL'FGAT B.E., On the bounded two-body of variable mass problem.Astron. Zh. Vol.34, No.4, 1957.

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